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INTERACTION OF STEADY SHEAR FLOWS OF A BAROTROPIC LIQUID

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A mathematical model describing plane-parallel vortex flows of a barotropic liquid with a free boundary in a long-wave approximation is considered. For a particular class of solutions, the solvability of the problem of decay of an initial discontinuity of small amplitude is demonstrated and an algorithm of solution is proposed.

The conditions of hyperbolicity of the system of equations for long waves propagating in a barotropic liquid layer are obtained by Teshukov [1]. A mathematical model for a hydraulic jump on a shear flow of a barotropic liquid is studied by Teshukov [2]. The properties of the simple waves corresponding to the characteristic velocities of a discrete spectrum are examined by Teshukov [3] and Elemesova [4]. In [5], a solution of the problem of decay of an arbitrary discontinuity for the system of equations of long waves propagating in an eddying incompressible liquid is proved to exist and an algorithm of determining the wave configurations that arise is proposed.

1. Formulation of the Problem. We consider the initial boundary-value problem

$$
\begin{gathered}
u_t + uu_x + vu_y + \rho^{-1}p_x = 0, \quad \rho^{-1}p_y = -g, \quad \rho_t + u\rho_x + v\rho_y + \rho(u_x + v_y) = 0, \quad u(x,y,0) = U_0(x,y), \\
h(x,0) = h_0(x), \quad \rho = \rho(p), \quad \rho' > 0, \quad 0 \leqslant y \leqslant h(x,t), \quad -\infty \leqslant x \leqslant +\infty, \quad t \geqslant 0, \qquad (1.1) \\
y = 0\colon \ v(x,0,t) = 0, \qquad y = h(x,\ t)\colon \ h_t + u(x,h(x,t),t)h_x = v(x,h(x,t),t),
\end{gathered}
$$

which describes vortex flow of a barotropic liquid with a free boundary in a shallow-water approximation. Here ρ is the density, u and v are the velocity-vector components, h is the free-surface equation, and p is the pressure. Integrating the second equation of (1.1), which describes the hydrostatic pressure distribution with depth

$$p_y = -\rho(p)g \qquad [p(x,h(x,t),t) = p_0],$$

and the continuity equation, we obtain the relation [1]

$$p = f(g(h-y)), \qquad \rho = f'(g(h-y)), \qquad v = -\rho^{-1}\int_0^y (\rho_t + (u\rho)_x)\,dy', \qquad (1.2)$$

where the function f is obtained by inversion of the relation $g(h-y) = \int_{p_0}^{p} \rho(s)^{-1}ds$. If the functions u, $h(x,t)$ are known, the quantities v, p, and ρ are found from (1.2).

The conditions of hyperbolicity for system (1.1) were studied in [1], where it is shown that system (1.1) has discrete and continuous spectra of characteristic velocities. The characteristic directions of the discrete

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spectrum $dx/dt = k_i$ are given by the following equation in the variables x, y, and t:

$$F(k_i) = R - g \int_0^{h(x,t)} (u(x,y,t) - k_i)^{-2} f'(g(h(x,t) - y))\, dy = 0 \quad [R = \rho(\eta), \quad \eta = f(gh)]. \tag{1.3}$$

For any x and t, Eq. (1.3) has only two substantial roots such that $k_1 < \min u$ and $k_2 > \max u$. The continuous spectrum occupies the entire range of the function u for fixed x and t.

In (1.1), from the independent variables x, y, and t, we convert to the new independent variables x, p, and t. The new unknown quantities are u, $\tau = p_t + up_x + vp_y$, y, and η. [The function $p = \eta(x,t)$ specifies the pressure distribution at the bottom.] The unknown boundary $y = h(x,t)$ in the variables x and p becomes known ($p = p_0$), and the boundary conditions takes the form

$$\tau = 0 \quad \text{for} \quad p = p_0. \tag{1.4}$$

The known boundary $y = 0$ in the new variables becomes unknown, and at $p = \eta(x,t)$, the following relation should hold:

$$\eta_t + u(x,\eta,t)\eta_x = \tau(x,\eta,t). \tag{1.5}$$

System (1.1) in the new variables becomes

$$u_t + uu_x + \tau u_p + R(\eta)^{-1}\eta_x = 0, \quad u_x + \tau_p = 0, \quad p_0 \leqslant p \leqslant \eta(x,t), \quad -\infty \leqslant x \leqslant +\infty, \quad t \geqslant 0. \tag{1.6}$$

If the flow is vortex-free, then $u_p \equiv 0$ and system (1.6) reduces to an analog in one-dimensional gas dynamics [1] [in the long-wave approximation, the vorticity ω is equal to $-u_y$ in the variables (x,p) — $\rho g u_p$].

Differentiating the first equation of (1.6) with respect to p and taking into account the second equation, we obtain the relation

$$(u_p)_t + u(u_p)_x + \tau(u_p)_p = 0,$$

whence it follows that (1.6) admits a particular class of solutions with constant or piecewise-constant quantity u_p. We consider flow with constant $u_p = \Omega = \text{const}$. Integrating this relation with respect to p from p_0 to η, we have $\eta = p_0 + (u_2 - u_1)/\Omega$, where $u_1 = u(x, p_0, t)$ and $u_2 = u(x,\eta,t)$. The horizontal and vertical components of the velocity vector are equal to $u = \Omega(p - p_0) + u_1$, $\tau = -u_{1x}(p - p_0)$, respectively. For the functions u_1 and u_2, from (1.6), we obtain the system

$$u_{1t} + u_1u_{1x} + (u_2 - u_1)_x(R\Omega)^{-1} = 0, \qquad u_{2t} + u_2u_{2x} + (u_2 - u_1)_x(R\Omega)^{-1} = 0. \tag{1.7}$$

The secular equation of system (1.7) has two roots

$$k_{1,2} = \left(u_2 + u_1 \mp \sqrt{(u_2 - u_1)^2 + 4(u_2 - u_1)R^{-1}\Omega^{-1}}\right)/2, \tag{1.8}$$

and the relations for the characteristics are converted to the Riemann invariants

$$r_{it} + k_i r_{ix} = 0, \quad r_{1,2} = u_1 - \int_0^{(u_2-u_1)/\Omega} \frac{2}{R(s+p_0)(\Omega s \pm \sqrt{\Omega^2 s^2 + 4s/(R(s+p_0))}\,)}\, ds. \tag{1.9}$$

We consider a two-layer flow in which the layer $p_0 \leqslant p \leqslant \eta$ is divided by the contact surface $p = \delta$ into two sublayers, in each of which the value of u_p is constant:

$$u_p = \begin{cases} \Omega_0, & p_0 \leqslant p \leqslant \delta(x,t), \\ \Omega, & \delta(x,t) \leqslant p \leqslant \eta(x,t). \end{cases}$$

At the interface $p = \delta$, the velocity vector (u, τ) is considered continuous. The horizontal velocity component has the form $u = \Omega_0(p - p_0) + u_1(x,t)$ at $p_0 \leqslant p \leqslant \delta(x,\ t)$ and $u = u_2(x,t) - \Omega(\eta - p)$ at $\delta \leqslant p \leqslant \eta$. The horizontal velocity component at the interface $p = \delta(x,\ t)$ is designated by $u_0 = \Omega_0(\delta - p_0) + u_1 = u_2 - \Omega(\eta - \delta)$. The thicknesses of the lower and upper layers are related to u_0, u_1, and u_2 by

$$\delta - p_0 = (u_0 - u_1)/\Omega_0, \qquad \eta - \delta = (u_2 - u_0)/\Omega. \tag{1.10}$$

From system (1.6) it follows that the vector $U = (u_2, u_1, u_0)^{\perp}$ (the superscript $\perp$ denotes transposition) should satisfy the equations

$$U_t + AU_x = 0, \quad A = \begin{pmatrix} u_2 + (R\Omega)^{-1} & -(R\Omega_0)^{-1} & R^{-1}(1/\Omega_0 - 1/\Omega)^{-1} \\ (R\Omega)^{-1} & u_1 - (R\Omega_0)^{-1} & R^{-1}(1/\Omega_0 - 1/\Omega)^{-1} \\ (R\Omega)^{-1} & -(R\Omega_0)^{-1} & u_0 + R^{-1}(1/\Omega_0 - 1/\Omega)^{-1} \end{pmatrix}. \tag{1.11}$$

The vertical velocity components in the lower and upper layers are given by the formulas $\tau = -(p-p_0)u_{1x}(x,t)$ and $\tau = (\delta - p)(u_{2x} - \Omega\eta_x) - u_{1x}(\delta - p_0)$, respectively.

In studying the wave processes in two-layer flow, it is necessary to know the velocities of propagation of the characteristics. The secular equation for system (1.11) can be obtained using the general rule of determining characteristics or the secular equation for the general case (1.3):

$$F(k_i) = R + \frac{1}{\Omega(u_2 - k_i)} - \frac{1}{\Omega_0(u_1 - k_i)} + \frac{1}{u_0 - k_i}\left(\frac{1}{\Omega_0} - \frac{1}{\Omega}\right) = 0. \tag{1.12}$$

Determination of the roots k_i of Eq. (1.12) reduces to searching for roots of the third-degree polynomial. It is known that the first root $k_1 < \min u_i$ and the second root $k_2 > \max u_i$. Depending on Ω and Ω_0, there are eight cases of the relative position of u_i and the third root k_3:

$$\begin{array}{lll} 1) & 0 < \Omega < \Omega_0, & u_1 < k_3 < u_0 < u_2; \\ 2) & \Omega > \Omega_0 > 0, & u_1 < u_0 < k_3 < u_2; \\ 3) & 0 > \Omega > \Omega_0, & u_2 < u_0 < k_3 < u_1; \\ 4) & \Omega < \Omega_0 < 0, & u_2 < k_3 < u_0 < u_1; \\ 5) & 0 < \Omega,\ \Omega_0 < 0, & u_1 < k_3 < u_2 < u_0; \\ 6) & 0 < \Omega,\ \Omega_0 < 0, & u_2 < k_3 < u_1 < u_0; \\ 7) & \Omega < 0,\ \Omega_0 > 0, & u_0 < u_1 < k_3 < u_2; \\ 8) & \Omega < 0,\ \Omega_0 > 0, & u_0 < u_2 < k_3 < u_1. \end{array} \tag{1.13}$$

From (1.13) it follows that the root k_3 falls in the velocity ranges in the upper layer or the lower layer. Since flows with constant u_p form a particular class of solutions of system (1.6), which has a continuous spectrum of characteristic velocities occupying the entire range of the velocity u at fixed x and t, the root k_3 is a nonisolated eigenvalue.

We verify the condition of strong nonlinearity for the sets of characteristics $dx/dt = k_i$ (i = 1, 2, and 3). We recall that a set of characteristics is called strongly linear if $\nabla k_i \cdot \boldsymbol{l}_{ri} \neq 0$, where $\boldsymbol{l}_{ri}$ is the right eigenvector of the matrix A that corresponds to the eigenvalue k_i; $\nabla = (\partial/\partial u_2,\ \partial/\partial u_1,\ \partial/\partial u_0)$. It is easy to verify that $\boldsymbol{l}_{ri} = ((u_2 - k_i)^{-1},\ (u_1 - k_i)^{-1},\ (u_0 - k_i)^{-1})$ is the right eigenvector of the matrix A. Using Eq. (1.12), we obtain

$$\nabla k_i \cdot \boldsymbol{l}_{ri} = K_i^{-1} L_i, \quad L_i = R'R + \frac{1}{\Omega(u_2 - k_i)^3} - \frac{1}{\Omega_0(u_1 - k_i)^3} + \frac{1}{(u_0 - k_i)^3}\left(\frac{1}{\Omega_0} - \frac{1}{\Omega}\right),$$

$$K_i = \frac{1}{\Omega(u_2 - k_i)^2} - \frac{1}{\Omega_0(u_1 - k_i)^2} + \frac{1}{(u_0 - k_i)^2}\left(\frac{1}{\Omega_0} - \frac{1}{\Omega}\right).$$

Since all roots of Eq. (1.12) are different, $K_i = F'_k(k_i) \neq 0$ and the sign of the function L_i determines whether the set of characteristics $dx/dt = k_i$ is strongly linear or not. We examine the sign of L_i for each set of characteristics. If the function $R = R(\eta)$ satisfies the condition [2]

$$3R - (\eta - p_0)R' > 0, \tag{1.14}$$

the functions L_1 and L_2 have a fixed sign. Indeed, for the roots k_1 and k_2 of Eqs. (1.12), the quantities $u_i - k_1$ and $u_i - k_2$ are positive or negative, and, hence, $(u_1 - k_i)(u_0 - k_i) > 0$ and $(u_0 - k_i)(u_2 - k_i) > 0$. In addition,

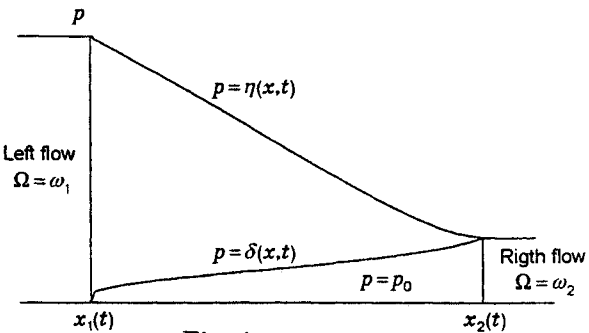

Fig. 1

$(u_0 - u_1)/\Omega_0 > 0$ and $(u_2 - u_0)/\Omega > 0$. Using the inequality $a^2 + b^2 \geqslant 2ab$, we obtain

$$L_i \leqslant RR' - \frac{3(u_2 - u_0)}{\Omega(u_2 - k_i)^2(u_0 - k_i)^2} - \frac{3(u_1 - u_0)}{\Omega_0(u_1 - k_i)^2(u_0 - k_i)^2} \qquad (i = 1,\ 2). \tag{1.15}$$

From Eq. (1.12) and the Cauchy-Bunyakovskii inequality it follows that

$$R \leqslant \left(\frac{u_2 - u_0}{\Omega} + \frac{u_0 - u_1}{\Omega_0}\right)^{1/2}\left(\frac{u_2 - u_0}{\Omega(u_2 - k_i)^2(u_0 - k_i)^2} + \frac{u_0 - u_1}{\Omega_0(u_0 - k_i)^2(u_1 - k_i)^2}\right)^{1/2} \quad (i = 1,\ 2).$$

From the last inequality, (1.10), and (1.15) we obtain

$$L_i \leqslant RR' - 3R^2(\eta - p_0)^{-1} = R(\eta - p_0)^{-1}((\eta - p_0)R' - 3R) < 0 \quad (i = 1,\ 2).$$

Let us consider the function L_3. We denote $\alpha = \Omega_0/\Omega$. Let condition (1.14) and the following condition [5] be satisfied:

$$1/2 < \alpha < 2. \tag{1.16}$$

We show that, in this case, the set of characteristics that corresponds to the root k_3 also satisfies the condition of strong nonlinearity. Using (1.14), we find that the function L_3 satisfies the inequality $L_3 \leqslant R^3\Omega_0^2(\psi_1 + \psi_2)$, where $\psi_1 = 3(\alpha/z_2 - 1/z_1 + (1-\alpha)/z_0)^{-1}$, $\psi_2 = \alpha z_2^3 - z_1^3 + (1-\alpha)z_0^3$, and $z_i = (R\Omega_0(u_i - k_3))^{-1}$, where $i = 0$, 1, and 2. The variables z_i are related by the secular equation $\chi = 1 + \alpha z_2 - z_1 + (1-\alpha)z_0 = 0$. Each case of variation of the parameters u_i and k_3 (1.13) corresponds to a particular range of the variables z_i. For example, case 1 corresponds to the range $z_1 < 0$, $z_0 > z_2 > 0$, $\alpha > 1$, and case 2 to the range $z_2 > 0$, $z_0 < z_1 < 0$, $0 < \alpha < 1$. The function $\psi = \psi_1 + \psi_2$ is tested for a conditional extremum subject to the condition $\chi = 0$ in each of the ranges of the variables z_i. Next, the behavior of the function ψ is examined at the boundaries of the ranges. As a result, it is shown that ψ is a negative function only in cases 1–4 with α satisfying condition (1.16). Below, we assume that conditions (1.14) and (1.16) are satisfied.

2. Existence of a Wave of Flow Interaction. We note that system (1.6) admits a solution of the form $u = u(p)$, $\tau = 0$, and $\eta = \eta_0 = \text{const}$, which in the initial variables x, y, and t describes a steady shear flow: $u = u(y)$, $v = 0$, and $h = h_0 = \text{const}$.

We consider the auxiliary problem of the interaction of two steady shear flows characterized by the constant quantity $u_p = \Omega$. Let, at $x < x_1(t)$ and $x > x_2(t)$, the flow be a shear flow (Fig. 1); $\Omega = \omega_1$ for the left flow $[x < x_1(t)]$ and $\Omega = \omega_2$ for the right flow $[x > x_2(t)]$. At $x_1(t) \leqslant x \leqslant x_2(t)$, on the left of the contact surface $p = \delta(x,t)$ there is the flow region with constant $u_p = \omega_1$, and on the right of it there is the flow region with constant $u_p = \omega_2$. It is assumed that at the boundaries $x = x_1(t)$ and $x = x_2(t)$ the conditions of continuous joining to shear flows are satisfied, and at the interface between the liquids $p = \delta(x,t)$, the continuity condition for the velocity (u, τ) and the condition $\delta_t + u\delta_x = \tau$ are satisfied.

We show that with certain limitations on the shear-flow velocity and satisfaction of inequalities (1.14) and (1.16) there is a simple wave — a wave of flow interaction — that satisfies the conditions listed above.

Since u_p is a piecewise-constant function at $x_1(t) \leqslant x \leqslant x_2(t)$, this region flow is described by system

(1.11). From (1.10) it follows that as $\delta \to p_0$, the velocity at the interface u_0 tends to the velocity u_1 ($p = p_0$), and as $\delta \to \eta$, we have $u_0 \to u_2$ ($p = \eta$). From the secular equation it follows that $u_0 \to k_3$ as $u_0 \to u_1$ ($\delta \to p_0$) and as $u_0 \to u_2$ ($\delta \to \eta$). Hence, the boundaries of the interaction region move with characteristic velocity $k = k_3$, and the flow region $x_1 \leqslant x \leqslant x_2$ should be described using the simple wave that corresponds to the root k_3 of Eqs. (1.12). From (1.11) we obtain the following equations for the simple wave $u_i = u_i(k(x,t))$, $k = k_3$:

$$(u_1 - k)u_1' = -\eta' R(\eta)^{-1}, \quad (u_2 - k)u_2' = -\eta' R(\eta)^{-1}, \quad (u_0 - k)u_0' = -\eta' R(\eta)^{-1}. \tag{2.1}$$

The equation for $\eta(k)$ is obtained by differentiating (1.12) with respect to k:

$$\eta'/R(\eta) = -K_3(L_3)^{-1}. \tag{2.2}$$

For system (2.1), (2.2), the secular equation (1.12) is an integral by construction.

We clarify the qualitative behavior of the simple-wave parameters. The derivative $\eta'(k)$ has no singularities since the strong nonlinearity condition is satisfied. The function L_3 is always negative, and the function K_3 is written as

$$K_3 = F'(k_3) = -(k_3 - k_1)(k_3 - k_2)[(u_2 - k_3)(u_1 - k_3)(u_0 - k_3)]^{-1}.$$

Hence, in cases 1–4 of the relative position of u_i and k_3 (1.13), the derivatives u_i' and η' have fixed sign:

$$\begin{aligned}
&1)\ \ \eta' < 0,\ \ u_2' > 0,\ \ u_0' > 0,\ \ u_1' < 0 \quad (\Omega_0 = \omega_2 > \Omega = \omega_1 > 0);\\
&2)\ \ \eta' > 0,\ \ u_2' < 0,\ \ u_0' > 0,\ \ u_1' > 0 \quad (0 < \Omega_0 = \omega_2 < \Omega = \omega_1);\\
&3)\ \ \eta' > 0,\ \ u_2' > 0,\ \ u_0' > 0,\ \ u_1' < 0 \quad (\Omega_0 = \omega_1 < \Omega = \omega_2 < 0);\\
&4)\ \ \eta' < 0,\ \ u_2' < 0,\ \ u_0' > 0,\ \ u_1' > 0 \quad (0 > \Omega_0 = \omega_1 > \Omega = \omega_2).
\end{aligned}$$

For definiteness, we consider case 1. In this case, the thickness of the lower layer δ increases with increase in k [$\delta' = (u_0' - u_1')/\omega_2 > 0$] and k varies in the range $u_1^{(l)} \leqslant k \leqslant u_2^{(r)}$. Below, the superscript l corresponds to the flow at the left [$x < x_1(t)$], and the superscript r to the flow on the right [$x > x_2(t)$].

Let us prove the existence of the simple wave of flow interaction by obtaining *a priori* estimates of the solution.

Lemma. *Let u_1, u_2, u_0, and η be a solution of system* (2.1), (2.2). *Then, the following estimates hold:*

$$\begin{gathered}
\frac{\omega_2\omega_1(\eta - \delta)(\delta - p_0)}{R(\eta)\omega_2^2\omega_1^2(\eta - \delta)(\delta - p_0) + \omega_1\omega_2(\eta - \delta) + \omega_2^2(\delta - p_0)} \leqslant |u_0 - k| \leqslant \sqrt{\frac{\eta - \delta}{R(\eta)}},\\
\omega_1(\delta - p_0) \leqslant |u_1 - k| \leqslant \omega_2(\delta - p_0), \quad \omega_1(\eta - \delta) \leqslant |u_2 - k| \leqslant \omega_1(\eta - \delta) + \sqrt{\frac{\eta - \delta}{R(\eta)}}.
\end{gathered} \tag{2.3}$$

Proof. In case 1 of (1.13), the inequalities

$$0 < u_0 - k < u_2 - k, \qquad \omega_2 > \omega_1, \qquad u_1 - k < 0 \tag{2.4}$$

are valid. Then, from the equality

$$u_2 - k = u_2 - u_0 + u_0 - k \tag{2.5}$$

it follows that

$$u_2 - k \geqslant u_2 - u_0. \tag{2.6}$$

From the equality $u_0 - u_1 = u_0 - k + k - u_1$, we obtain the upper bound

$$|u_1 - k| \leqslant u_0 - u_1. \tag{2.7}$$

From the secular Eq (1.12) it follows that

$$-\frac{u_2 - u_0}{\omega_1(u_2 - k)(u_0 - k)} - \frac{u_0 - u_1}{\omega_2(u_0 - k)(u_1 - k)} < 0.$$

Hence, by virtue of (2.4), (2.6), we have

$$|u_1 - k| > \frac{\omega_1(u_0 - u_1)}{\omega_2(u_2 - u_0)}(u_2 - k) > \omega_1\omega_2^{-1}(u_0 - u_1). \tag{2.8}$$

Since $(u_0 - u_1)(\omega_2(u_0 - k)(u_1 - k))^{-1} < 0$, from (1.12) we obtain $R - (u_2 - u_0)(\omega_1(u_0 - k)(u_2 - k))^{-1} < 0$. The last inequality and (2.4) yield the upper bound

$$(u_0 - k)^2 \leqslant (u_0 - k)(u_2 - k) \leqslant (u_2 - u_0)(R\omega_1)^{-1}. \tag{2.9}$$

By virtue of (2.6) and (2.8), from (1.12) we obtain

$$\frac{1}{u_0 - k}\left(\frac{1}{\omega_1} - \frac{1}{\omega_2}\right) = R + \frac{1}{\omega_1(u_2 - k)} - \frac{1}{\omega_2(u_1 - k)} \leqslant R + \frac{1}{\omega_1(u_2 - u_0)} + \frac{1}{\omega_1(u_0 - u_1)},$$

whence, we have the lower bound

$$u_0 - k \geqslant \frac{\omega_2 - \omega_1}{\omega_2} \frac{(u_2 - u_0)(u_0 - u_1)}{R\omega_1(u_2 - u_0)(u_0 - u_1) + u_2 - u_1}. \tag{2.10}$$

From (2.5) and (2.9), we obtain the upper bound

$$u_2 - k \leqslant u_2 - u_0 + \sqrt{(u_2 - u_0)/(R\omega_1)}. \tag{2.11}$$

The statement of the lemma follows from (1.10) and inequalities (2.6)–(2.11). The lemma is proved.

From the *a priori* estimates (2.3) it follows that in any interval $\delta \in [\sigma, A]$ $(p_0 < \sigma \leqslant \delta \leqslant A < \eta)$, the quantities $|u_i - k|$ remain bounded: $M_1(\sigma, A, p_0, \eta) < |u_i - k| < M_2(\sigma, A, p_0, \eta)$. The quantity $u_1 - k$ vanishes only at $\delta = p_0$ (the boundary point $k = u_1$), and the quantity $u_2 - k$ vanishes only at $\delta = \eta$ (the boundary point $k = u_2$). The quantity $u_0 - k$ can vanish only at the boundary points $\delta = \eta$ and $\delta = p_0$.

We consider the behavior of solution (2.1) at the boundary points. As $k \to u_1$, $u_1 \to u_0$, and $\delta \to p_0$ from (1.12) we obtain $\lim\limits_{k \to u_1} [(u_0 - k)/(u_1 - k)] = 1 - \alpha$. From (2.2) it follows that in the neighborhood of the point $k = u_1^{(l)}$, the function η'/R is written as

$$\frac{\eta'}{R} = -\frac{u_0 - k}{2 - \alpha} + O((u_0 - k)^2).$$

The limits in the neighborhood of the point $k = u_2^{(r)}$ are found in the same manner:

$$\lim_{k \to u_2} \frac{u_0 - k}{u_2 - k} = (\alpha - 1)\alpha^{-1}, \qquad \frac{\eta'}{R} = -\frac{u_0 - k}{2\alpha - 1} + O((u_0 - k)^2).$$

Thus, the derivatives of u_i have finite limits at the boundary points $k = u_1^{(l)}$ and $k = u_2^{(r)}$:

$$\begin{gathered} u_1'(u_1^{(l)}) = (1 - \alpha)(2 - \alpha)^{-1}, \qquad u_2'(u_1^{(l)}) = 0, \qquad u_0'(u_1^{(l)}) = (2 - \alpha)^{-1}, \\ u_1'(u_2^{(r)}) = 0, \qquad u_2'(u_2^{(r)}) = (\alpha - 1)(2\alpha - 1)^{-1}, \qquad u_0'(u_2^{(r)}) = \alpha(2\alpha - 1)^{-1}. \end{gathered} \tag{2.12}$$

Considering the left flow known, we determine the right flow, which can be related to the specified flow by a simple wave of interaction. The condition of joining to the left flow ($\delta = p_0$) is

$$u_0 = u_1^{(l)}, \qquad u_1 = u_1^{(l)}, \qquad u_2 = u_2^{(l)} \quad \text{for} \quad k = u_1^{(l)}. \tag{2.13}$$

From (2.12) it follows that the Cauchy problem (2.1), (2.2), (2.13) has a single solution in the neighborhood of the boundary point $k = u_1^{(l)}$. The possibility of continuation of the solution follows from the *a priori* estimates [by virtue of (2.3), we obtain a uniform estimate for the right sides of system (2.1), (2.2)]. Since in the interaction wave, the thickness of the lower layer δ increases with increase in k, the solution continues to the point $\delta = \eta$ [$k = k_*$ and $u_2(k_*) = k_*$]. The value of k_* is determined in the course of solution of the problem. Then, the state that can be related to the left state by a simple interaction wave is given by the relations

$$u_1^{(r)} = u_1(k_*, u_1^{(l)}, u_2^{(l)}) = \varphi_1(u_1^{(l)}, u_2^{(l)}), \qquad u_2^{(r)} = k_*(u_1^{(l)}, u_2^{(l)}) = \varphi_2(u_1^{(l)}, u_2^{(l)}). \tag{2.14}$$

Thus, a simple wave of flow interaction is proved to exist, and necessary conditions for the existence of this wave are obtained.

3. Problem of Discontinuity Decay. We consider the problem of decay of an initial discontinuity for system (1.1):

$$(u,h)\Big|_{t=0} = \begin{cases} (u^l(y),\, h^l) & \text{at} \quad x<0, \\ (u^r(y),\, h^r) & \text{at} \quad x>0. \end{cases} \tag{3.1}$$

The initial data (3.1) describe steady shear flows on the left and right of the boundary of the discontinuity ($x=0$). Here $u^l(y)$ and $u^r(y)$ are known functions and h^l and h^r are specified constants.

In the variables x and p, the initial data of the problem of decay of an arbitrary discontinuity (3.1) have the form

$$(u,\eta)\Big|_{t=0} = \begin{cases} (u^l(p),\, \eta_1) & \text{for} \quad x<0, \\ (u^r(p),\, \eta_2) & \text{for} \quad x>0. \end{cases} \tag{3.2}$$

Next, we consider the initial data (3.2) with constant $u_p^l = \omega_1$ and $u_p^r = \omega_2$. These conditions simplify the problem and they follow from the property that the quantities $\omega/(\rho g) = u_p$ are conserved with passage through the fronts of small-amplitude discontinuities and regions of simple waves [2, 4].

Since the equations, the initial data, and the boundary conditions are invariant with respect to uniform extension of the variables x and t, we seek a solution of the problem of decay of an arbitrary discontinuity (1.8) in the class of self-similar solutions $u = u(x/t,\ p)$, $\eta = \eta(x/t,p)$, and $\tau = t^{-1}\tau(x/t,p)$.

For the single-layer flow region, the equations of motion (1.7) are written as

$$(\eta - p_0)_t + (u_c(\eta - p_0))_x = 0, \quad ((\eta - p_0)u_c)_t + (u_c^2(\eta - p_0))_x + (P(\eta - p_0))_x = 0, \tag{3.3}$$

where $u_c = (u_2 + u_1)/2$ and $P(\eta - p_0) = \displaystyle\int_0^{\eta - p_0} sR(p_0 + s)^{-1}\, ds + \omega_i^2(\eta - p_0)^3/12$. For Eqs. (3.3), the initial data of the problem of discontinuity decay have the form

$$(u_c,\eta)\Big|_{t=0} = \begin{cases} (u_{c1},\, \eta_1) & \text{for} \quad x<0, \\ (u_{c2},\, \eta_2) & \text{for} \quad x>0 \end{cases} \tag{3.4}$$

(u_{ci} and η_i are specified constants).

System (3.3) is an analog of one-dimensional gas-dynamic equations. The relation $P(\eta - p_0)$ describes the equation of state. It is known [6] that the properties of solutions of gas-dynamic equations depend on the properties of the function $\varphi(V) = P(V)$ (V is the specific volume). If $\varphi_V < 0$ and $\varphi_{VV} > 0$, centered waves that arise upon decay of initial discontinuities are always rarefaction waves and the discontinuities are compression shock waves (in our formulation of the problem, they are hydraulic jumps of level rise). In this case, a right wave and a left wave always form. For the equation of state $P(\eta - p_0)$, the condition $\varphi_V < 0$ is always satisfied and the convexity condition is satisfied by virtue of condition (1.14).

Following the algorithm of solving the gas-dynamic problem of discontinuity decay, we define a set of states (u_c,η) that can be related to the initial state (u_{c0},η_0) by a simple wave or a strong discontinuity. The discontinuity relations for (3.3) have the form

$$[(\eta - p_0)(u_c - D)] = 0, \qquad [(\eta - p_0)u_c(u_c - D) + P] = 0, \tag{3.5}$$

where $[f] = f^+ - f^-$ is the jump of the function on the discontinuity $x'(t) = D$. Let u_{c0} and η_0 be the flow parameters before the jump. Then, from (3.5) we find u_c and η behind the front:

$$u_c - u_{c0} = \pm\sqrt{(P - P_0)((\eta_0 - p_0)^{-1} - (\eta - p_0)^{-1})} \tag{3.6}$$

(the plus or minus is selected for the right or left wave, respectively).

The states that can be related to $(u_{c0},\, \eta_0)$ by a simple wave are defined by the relations

$$r_2 = r_{20} = \text{const}, \qquad r_1 = r_{10} = \text{const}. \tag{3.7}$$

The first relation specifies a left simple rarefaction wave (the speed of propagation of the wave is $k = k_1 < \min u_i$), and the second relation defines a right wave ($k = k_2 > \max u_i$). We convert (3.7) to the relations between $\eta - p_0$ and u_c. Using (1.9) and the equalities $u_1 = u_c - \omega_i(\eta - p_0)/2$ and $u_2 = u_c + \omega_i(\eta - p_0)/2$, we write (3.7) in the form

$$u_c - u_{c0} = \begin{cases} \Phi_1(\eta - p_0) - \Phi_1(\eta_0 - p_0) + \omega_i(\eta - \eta_0)/2, & r_1 = r_{10}, \\ \Phi_2(\eta - p_0) - \Phi_2(\eta_0 - p_0) + \omega_i(\eta - \eta_0)/2, & r_2 = r_{20}, \end{cases} \tag{3.8}$$

$$\Phi_i = \int\limits_0^{\eta - p_0} \frac{2}{R(s+p_0)(\omega_i s \pm \sqrt{\omega_i^2 s^2 + 4s/(R(s+p_0))}\,)}\,ds, \qquad i = 1,\ 2.$$

The right sides of (3.6) and (3.8) are monotonic functions that vanish for $\eta = \eta_0$.

For specified values (u_{c1}, η_1), which will be called the left initial state, the solution for the wave (rarefaction or shock), $u_c^{(l)} = u_{c1} + U^l(\eta_1, \eta^{(l)})$ is a function of one variable $\eta^{(l)}$. The function U^l is given by the second relation of (3.8) for $\eta^{(l)} < \eta_1$ and by formula (3.6) with the minus sign for $\eta^{(l)} > \eta_1$. For the state (u_{c2}, η_2), which will be called the right state, the solution is defined as a function of the variable $\eta^{(r)}$: $u_c^{(r)} = u_{c2} + U^r(\eta_1, \eta^{(r)})$. The function U^r is given by the first relation of (3.8) for $\eta^{(r)} < \eta_2$ and formula (3.6) with the plus sign for $\eta^{(r)} > \eta_2$. In contrast to the gas-dynamic problem, the states behind the transmitted waves must satisfy conditions (2.14) rather than the condition of coincidence of velocities and pressure. We bring (2.14) to the form

$$\begin{gathered} u_c^{(r)} = (\varphi_1(u_1^{(l)}, u_2^{(l)}) + \varphi_2(u_1^{(l)}, u_2^{(l)}))/2 = U_1(u_c^{(l)},\ \eta^{(l)}), \\ \eta^{(r)} - p_0 = (\varphi_2(u_1^{(l)}, u_2^{(l)}) - \varphi_1(u_1^{(l)}, u_2^{(l)}))/\omega_2 = U_2(u_c^{(l)}, \eta^{(l)}), \end{gathered} \tag{3.9}$$

where $u_c^{(r)} = (u_2^{(r)} + u_1^{(r)})/2$, $\eta^{(r)} - p_0 = (u_2^{(r)} - u_1^{(r)})/\omega_2$, $u_2^{(l)} = u_c^{(l)} + \omega_1(\eta^{(l)} - p_0)/2$, and $u_1^{(l)} = u_c^{(l)} - \omega_1(\eta^{(l)} - p_0)/2$.

Solution of the problem of decay of an initial discontinuity (3.3) and (3.4) reduces to solution of the following system of four equations for the four unknowns $u_c^{(l)}$, $\eta^{(l)}$, $u_c^{(r)}$, and $\eta^{(r)}$:

$$\begin{gathered} \Psi_1 \equiv u_c^{(l)} - u_{c1} - U^l(\eta_1, \eta^{(l)}) = 0, \qquad \Psi_2 \equiv u_c^{(r)} - u_{c2} - U^r(\eta_2, \eta^{(r)}) = 0, \\ \Psi_3 \equiv u_c^{(r)} - U_1(u_c^{(l)}, \eta^{(l)}) = 0, \qquad \Psi_4 \equiv \eta^{(r)} - U_2(u_c^{(l)}, \eta^{(l)}) = 0. \end{gathered} \tag{3.10}$$

If the left and right shear flows coincide, i.e., $u_{c1} = u_{c2}$, $\eta_1 = \eta_2$, $\omega_1 = \omega_2$, then $\alpha = 1$ and the interaction wave degenerates into a shear flow:

$$u = \omega_1(p - p_0) + u_1^{(l)}, \quad \eta = \eta_1, \quad u_1 = u_1^{(l)}, \quad u_2 = u_2^{(l)}, \quad k = u_0 = u_0^0 \quad (u_1^{(l)} \leqslant k \leqslant u_2^{(l)}). \tag{3.11}$$

Condition (2.14) becomes the condition of coincidence of the left and right shear flows: $u_c^{(r)} = u_c^{(l)}$ and $\eta^{(r)} = \eta^{(l)}$, and system (3.10) is compatible.

Let the left and right states be close. Then, $\omega_2 = (1+\varepsilon)\omega_1$ and $\alpha = 1 + \varepsilon$, where ε is a small parameter ($\varepsilon > 0$). We consider small perturbations of the shear flow (3.11):

$$\eta = \eta_1 + \varepsilon\eta^1, \quad u_1 = u_1^{(l)} + \varepsilon u_1^1, \quad u_2 = u_2^{(l)} + \varepsilon u_2^1, \quad u_0 = u_0^0 + \varepsilon u_0^1 = k^0 + \varepsilon u_0^1, \quad k = k^0 + \varepsilon k^1.$$

We linearize system (2.1), retaining only terms that are linear in ε, and refer all perturbations to the unperturbed level $k = u_0^0 = k^0$. We obtain the system

$$\begin{gathered} (u_1^1)' = -(\eta^1)' R_0^{-1}(u_1^{(l)} - k)^{-1}, \quad (u_2^1)' = -(\eta^1)' R_0^{-1}(u_2^{(l)} - k)^{-1}, \quad u_0^1 = -(\eta^1)' R_0^{-1}, \\ R_0 = R(\eta_1) = \text{const}, \qquad k = u_0^0. \end{gathered} \tag{3.12}$$

Linearizing (1.9) with allowance for $\eta_1 = (u_2^{(l)} - u_1^{(l)})\omega_1^{-1}$, we have

$$\eta^1 = (u_2^1 - u_1^1)\omega_1^{-1} + (u_1^{(l)} - k)\omega_1^{-1}. \tag{3.13}$$

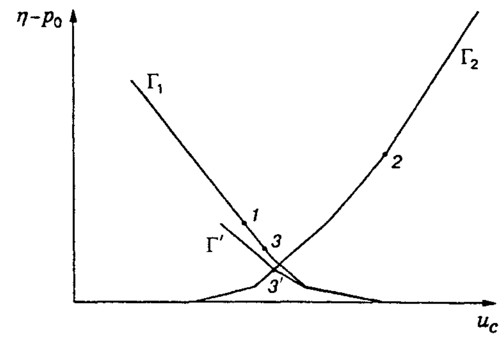

Fig. 2

The quantity k^1 is obtained from the linearized secular equation(1.12):

$$k^1 = u_0^1 - (R_0\omega_1 + (u_2^{(l)} - k)^{-1} - (u_1^{(l)} - k)^{-1})^{-1}.$$

Using (3.13) and eliminating η^1 from the first two equations (3.12), we obtain

$$(u_1^1)' = \frac{(u_2^{(l)} - k)/(R_0\omega_1)}{(u_2^{(l)} - k)(u_1^{(l)} - k) - (u_2^{(l)} - u_1^{(l)})/(R_0\omega_1)},$$
$$(u_2^1)' = \frac{(u_1^{(l)} - k)/(R_0\omega_1)}{(u_2^{(l)} - k)(u_1^{(l)} - k) - (u_2^{(l)} - u_1^{(l)})/(R_0\omega_1)}. \tag{3.14}$$

Since conditions (2.14) should be satisfied, we have $u_1^1(u_1^{(l)}) = u_2^1(u_1^{(l)}) = 0$. Integrating (3.14), we find the following solution of the linearized problem:

$$u_1^1 = (I_1(k) - I_1(u_1^{(l)}))(R_0\omega_1\Delta)^{-1}, \qquad u_2^1 = (I_2(k) - I_2(u_1^{(l)}))(R_0\omega_1\Delta)^{-1}.$$

Here $I_1(k) = \ln{(k_2 - k)^{u_1^{(l)} - k_2}(k - k_1)^{k_1 - u_1^{(l)}}}$ and $I_2(k) = \ln{(k_2 - k)^{u_2^{(l)} - k_2}(k - k_1)^{k_1 - u_2^{(l)}}}$, Δ is the discriminant, and k_1 and k_2 are roots of the quadratic equation $(u_2^{(l)} - k)(u_1^{(l)} - k) - (u_2^{(l)} - u_1^{(l)})/(R_0\omega_1) = 0$ $(k_2 > k_1)$. Then, the Jacobian of system (3.10) for $u_{c1} = u_{c2}$, $\eta_1 = \eta_2$, and $\omega_1 = \omega_2$ is not equal to zero, and according to the implicit function theorem, in the neighborhood of (u_{c1}, η_1) there is a unique solution of system (3.10).

We describe an algorithm of constructing a solution of the problem of decay of an arbitrary discontinuity. We construct plots of Γ_1 and Γ_2 of the possible transitions for the left and right initial states, indicated by points 1 and 2, respectively, in the plane $(u_c, \eta - p_0)$ (Fig. 2). The curve Γ_1 is given by the function $u_c = u_{c1} + U^{(l)}$, and the curve Γ_2 is given by the function $u_c = u_{c2} + U^{(r)}$. The states behind the transmitted wave fronts must be related by the existence conditions for an interaction wave (2.14). To find the point that specifies the state behind the front, we plot the curve Γ' given by the parametric equations (3.9), where u_{c1} and η_1 belong to the curve Γ_1. The state behind the transmitted waves for initial state 2 is specified by the point of intersection 3′ of the curves Γ_2 and Γ', and the state behind the wave front for initial state 1 is specified by point 3 (preimage of point 3′ on the curve Γ_1). The wave configuration is determined by analogy with gas dynamics: if the point of intersection 3′ (or 3) lies above the initial point 2 (or 1), a shock wave (hydraulic jump) propagates over the shear flow, and if it is below the initial point, a simple rarefaction wave propagates. The states behind the front are related by relation (2.14), and the solution of the problem of decay of an arbitrary discontinuity is completed by constructing the simple interaction wave. In contrast to [5], the existence conditions for a flow interaction wave (2.14) are not specified explicitly but are found from the solution of system (2.1), (2.2), which can be performed numerically.

We plot the curve Γ'. We select several points (u_{c1}^i, η_1^i) on the curve Γ_1. The values $u_{1i}^{(l)}$, $u_{2i}^{(l)}$ are uniquely determined from (u_{c1}^i, η_1^i) by formulas (3.9). Using $u_{1i}^{(l)}$ and $u_{2i}^{(l)}$ as Cauchy data, for each i, we obtain a solution $u_{1i}^{(r)}$, $u_{2i}^{(r)}$ of system (2.1), (2.2). The point set $(u_{ci}^{(r)}, \eta_i^{(r)})$ specifies the curve Γ' [the values of $u_{ci}^{(r)}$

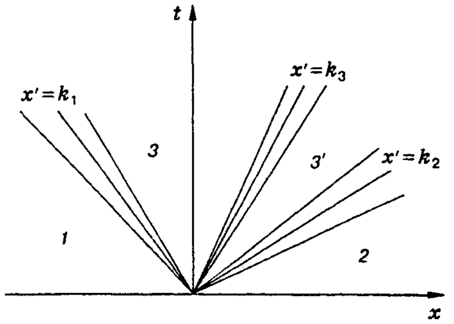

Fig. 3

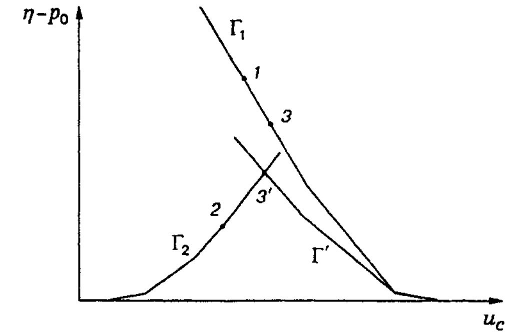

Fig. 4

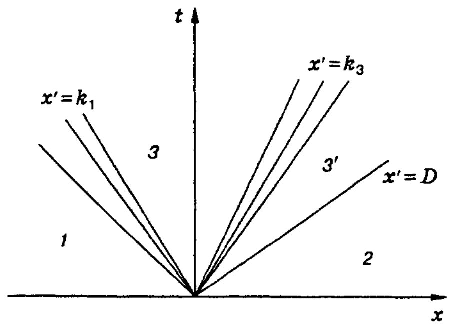

Fig. 5

and $\eta_i^{(r)}$ are obtained from (3.9)]. Figure 2 gives plots of Γ_1, Γ_2, Γ' for the left state $u_1 = 1$, $u_2 = 2$, and $\omega_1 = 2$ and the right state $u_1 = 2$, $u_2 = 5$, and $\omega_2 = 3$ [the polytropic equation of state $R(\eta) = \eta^{1/3}$]. The differential equations (2.1) and (2.2) are approximated by an explicit difference scheme [7]. In the neighborhood of the point $k = u_1 = 1$, the solution is determined from (2.11) and is then found by the second-order Runge–Kutta method [7]. In this case, simple rarefaction waves propagate over the shear flows. Figure 3 gives the pattern of characteristics in the plane (x, t) that corresponds to the wave configuration in Fig. 2. When the position of states 1 and 2 changes, the position of the curve Γ' and, hence, the wave configurations change.

Figure 4 shows the curves of Γ_1, Γ_2, and Γ' for the left state $u_1 = 1$, $u_2 = 7$, and $\omega_1 = 2$ and the right state $u_1 = 2$, $u_2 = 5$, and $\omega_2 = 3$ [the equation of state $R(\eta) = \eta^{1/3}$]. As follows from Fig. 4, a rarefaction wave propagates over state 1 and a hydraulic jump propagates over state 2. The pattern of characteristics corresponding to this wave configuration is given in Fig. 5.

Thus, let the left state 1 (u_{c1}, η_1) and the right state 2 (u_{c2}, η_2) be specified. The curves Γ_1 and Γ_2 of the possible transitions for states 1 and 2, respectively are plotted in the plane $(\eta - p_0, u_c)$ (see Fig. 2). Then, the curve Γ' is plotted by the curve Γ_1. Point 3′ with the coordinates $(u_c^{(r)}, \eta^{(r)})$ of intersection of the curves Γ' and Γ_2 specifies the state behind front that corresponds to state 2, and point 3 with the coordinates $(u_c^{(l)}, \eta^{(l)})$ (pre-image of point 3′ on the curve Γ_1) specifies the state behind front that corresponds to state 1. If point 3′ is above point 2 on the curve Γ_2, a hydraulic jump propagates over the shear flow; if point 3′ is below point 2, a simple wave of decreasing level propagates. Similarly, for condition 1: if point 3 lies above point 1 on the curve Γ_1, a hydraulic jump propagates, and if it is lower, a simple wave of decreasing level propagates. For the wave configuration shown in Fig. 2, a simple wave with characteristic velocity $k_2 > \max u$ propagates over the right flow [$k_2 = k_2(u_{c2}, \eta_2)$ is determined from formula (1.8) with the plus sign]. A simple wave with velocity $k_1 < \min u$ propagates over the left flow (k_1 is determined from (1.8) with the minus sign). In the regions $-\infty < x/t < k_1(u_{c1}, \eta_1)$ and $k_2(u_{c2}, \eta_2) < x/t < +\infty$ (see Fig. 3), the flow is a shear flow: $u_c = u_{c1}$, $\eta = \eta_1$ and $u_c = u_{c2}$, $\eta = \eta_2$, respectively. In the simple-wave region adjacent to state 1, the solution is given by the formulas $x/t = k_1(u_c, \eta)$ and $r_2 = r_2(u_{c1}, \eta_1)$, where $k_1(u_{c1}, \eta_1) \leqslant k_1 \leqslant k_1(u_c^{(l)}, \eta^{(l)})$. In the simple-wave region adjacent to state 2, the solution is given by the formulas $x/t = k_2(u_c, \eta)$, and $r_1 = r_1(u_{c2}, \eta_2)$, where $k_2(u_{c2}, \eta_2) \leqslant k_2 \leqslant k_2(u_c^{(r)}, \eta^{(r)})$. In Fig. 3, the flow in the regions $k_1(u_c^{(l)}, \eta^{(l)}) < x/t < u_1^{(l)}$ and $u_2^{(r)} < x/t < k_2(u_c^{(r)}, \eta^{(r)})$ is a shear flow. The state corresponding to points 3 and 3′ obeys conditions (2.14), i.e., among all possible states, we found two states that can be related by a simple interaction wave. Construction of the simple wave in the region $u_1^{(l)} < x/t < u_2^{(r)}$ completes the solution of the problem.

For the wave configuration shown in Fig. 4, the solution is constructed in the same manner. The velocity of the hydraulic jump D is obtained from the states ahead of and behind the discontinuity front using the first relation of (3.5). In our case, $D = ((\eta^{(r)} - p_0)u_c^{(r)} - (\eta_2 - p_0)u_{c2})(\eta^{(r)} - \eta_2)^{-1}$. We analyze the discontinuity relations. From (3.5) it follows that $u_c^{(r)} < D$, where $u_c^{(r)}$ is the mean velocity behind the discontinuity front. However, the maximum velocity $u_2^{(r)}$ may not satisfy this condition (in the case considered, $\omega_2 > 0$). Using relations (3.5) and (3.9), we write the condition $u_2^{(r)} = D$ in the form

$$\omega_2^2(\eta_2 - p_0) = 12((\xi - 1)^2(3\xi^2 + 2\xi + 1))^{-1} \int_1^{\xi} sR(p_0 + s)^{-1}\, ds \quad (\xi = (\eta^{(r)} - p_0)(\eta_2 - p_0)^{-1}). \tag{3.15}$$

As $\xi \to 1$, the function on the right side of (3.15) tends to $+\infty$ (the uncertainty 0/0 is uncovered by the L'Hospital rule), and as $\xi \to +\infty$, it tends to zero. Therefore, there is always at least one root $\xi = \xi_*$ of Eq. (3.13). If $\omega_2^2(\eta_2 - p_0) \to +\infty$, then $\xi_* \to 1$ and if $\omega_2^2(\eta_2 - p_0) \to 0$, then $\xi_* \to +\infty$. Hence, the maximum velocity behind the front $u_2^{(r)}$ remains lower than the velocity of the discontinuity D if the quantity $\omega_2^2(\eta_2 - p_0)$ is small. Therefore, the configuration with hydraulic jumps belongs to the case of small quantities $\omega_i^2(\eta - p_0)$. We note that for a wave configuration with rarefaction waves there are no such limitations.

If the curves Γ' and Γ_2 are not intersected, decay gives rise to two centered simple rarefaction waves behind which the flow "depth" is equal to zero.

Thus, it is proved that for small values of $\omega_i^2(\eta_i - p_0)$ and close states 1 and 2, the problem of decay of an arbitrary discontinuity has a unique solution.

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